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Matone's relation in the presence of gravitational couplings

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Abstract: The prepotential in $N = 2$ SUSY Yang-Mills theories enjoys remarkable properties. One of the most interesting is its relation to the coordinate on the quantum moduli space $u = \langle \mathrm{Tr}\, \varphi^2 \rangle$ that results into recursion equations for the coefficients of the prepotential due to instantons. In this work we show, with an explicit multi-instanton computation, that this relation holds true at arbitrary winding numbers. Even more interestingly we show that its validity extends to the case in which gravitational corrections are taken into account if the correlators are suitably modified. These results apply also to the cases in which matter in the fundamental and in the adjoint is included. We also check that the expressions we find satisfy the chiral ring relations for the gauge case and compute the first gravitational correction.

Keywords: Topological Field Theories, Nonperturbative Effects, Solitons Monopoles and Instantons.

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1. Introduction

The chiral algebra of operators $O_m = \mathrm{Tr}\,\varphi^m$ in $N=2$ globally supersymmetric Yang-Mills (SYM from now on) gauge theories in four space time dimensions has been the subject of intense research in recent years. $\varphi(x)$ is the complex scalar field in the $N=2$ supersymmetric multiplet. One of the most remarkable results is the relation between the expectation value $u = \langle \mathrm{Tr}\,\varphi^2\rangle$ [1] and the $N=2$ prepotential

$$u(a) = i\pi\left(\mathcal{F}(a) - \frac{1}{2}a\frac{\partial}{\partial a}\mathcal{F}(a)\right) \tag{1.1}$$

where a is the v.e.v. of the scalar field. $u(a) = \sum_{k=0}^{\infty}\mathcal{G}_k(\Lambda/a)^{4k}a^2$, in fact, obeys a non-linear differential equation that leads to a recursion relation among the $\mathcal{G}_k's$. In turn, expanding $\mathcal{F}(a) = \sum_{k=0}^{\infty}\mathcal{F}_k(\Lambda/a)^{4k}a^2$ and using (1.1) we obtain $\mathcal{G}_k = 2\pi i k\mathcal{F}_k$ i.e. the explicit expression of the prepotential. We consider the $N=2$ theory with SU(2) as underlying gauge group. The generalization of (1.1) for models with higher rank gauge group has also been considered [2]. It was argued in [3] that (1.1) is a consequence of the Ward identities of (broken) superconformal invariance.

Inverting the functional dependence, i.e., taking $a = a(u)$ instead of $u = u(a)$, one derives from (1.1) for $a(u)$ and $a_D(u) = \partial\mathcal{F}/\partial a$ the second order differential equation

$$\frac{\partial^2}{\partial u^2}a_D(u) + V(u)a_D(u) = 0 \tag{1.2}$$

with $V(u) = -a_D^{-1}(u)\partial^2 a_D(u)/\partial u^2 = a^{-1}(u)\partial^2 a(u)/\partial u^2$. (1.1) (or (1.2)) alone does not allow for a determination of $u(a)$ and hence $\mathcal{F}(a)$. The necessary complementing information can be extracted from the Seiberg-Witten curves [4]. This leads in the SU(2) case to the determination $V(u) = -1/[4(1-u^2)]$. Furthermore in [5] $u(a)$ was expressed in terms of theta functions. Both approaches, that of [1] and that of [5], give rise to recursion relations for the expansion coefficients of the prepotential $\mathcal{F}(a)$.

A more recent achievement is the derivation of relations for expectation values of operators in the quantum chiral ring [6]. Those have been obtained by exploiting equations deduced from the Konishi anomaly.

In this paper we want to undertake the computation of the expectation values of the operators O_m within the framework of microscopic instanton calculus. The basic references to rely on for this aim are [7, 8]. In the first of these references the first computation for O_2 was carried out. In the second the general instanton configuration with its associated zero modes (and the classical solution for the scalar field) has been constructed. We will also make use of the technique of equivariant localization [9] which, in the context of instanton calculus has been first advanced in [10, 11] and has been put on a solid ground by [12] who observed that the deformation of the instanton configuration into the non commutative realm is a convenient device for the resolution of the singularities of the corresponding moduli space. See also [15, 14] for a more mathematical oriented description. The localization employed is akin to that of [16] who evaluated the equivariant Euler character. The method is discussed at length in [17] and extended to supermanifolds in [18]. The final ingredient to be used is found in [13] where it was proposed to use an extended vector field in the localization procedure which is related to the unbroken U(1) symmetries and to space time rotations. The introduction of the latter gives rise to a discrete set of critical points and makes therefore the task of evaluating the localized integrals feasible. See also [19, 17] for related work. It was also suggested in [13] that the parameters of the space time rotations, to be called below ϵ_1, ϵ_2, have a physical meaning in the sense that they are associated with gravitational couplings.

We will also derive the above mentioned relations of the chiral operator ring and discuss some of the modifications due to gravitational couplings. We find in this way an independent verification of (1.1) which was partially checked in [7, 20]. More importantly we find a deformation of the scalar field, $\tilde{\varphi}$ which for $\epsilon_1, \epsilon_2 \to 0$ falls back to φ and preserves for $\epsilon_1, \epsilon_2 \neq 0$ the chiral property. Moreover (1.1) and (1.2) hold for $\tilde{\varphi}$ in presence of gravitational couplings. It remains as an important open problem to find the explicit expression for $V(u)$ for $\epsilon_1, \epsilon_2 \neq 0$.

The survival of (1.1) in presence of gravitational couplings may not be too surprising. In fact in [3] it was argued that the l.h.s. in (1.1) should emerge from a gauge invariant operator of canonical dimension two. With or without ϵ_1, ϵ_2 there is no other operator with this specification but O_2.

The plan of the paper is the following: in section 2 we give some preliminaries. In section 3 we discuss the classical solution for the scalar field in the presence of a deformation whose parameters ϵ_1, ϵ_2 will be introduced in due time. In section 4 we compute correlators of the type $\langle \mathrm{Tr}\, \tilde{\varphi}^m \rangle$ and the chiral ring relation [6] with its first gravitational correction.

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The results of section 3 and 4 are to be compared with those obtained in [21]. We stress that the object which satisfies the Matone's relation in presence of the parameters ϵ_1, ϵ_2 is $\tilde{\varphi}$ which is not a solution of the Euler-Lagrange equations of motion. In the appendix we discuss the meaning of computing vacuum expectation values in the non commutative case and compare with the commutative case using the semiclassical approximation.

2. Preliminaries

2.1 A brief reminder of the ADHM construction

The starting point in the ADHM construction of $\mathrm{SU}(n)$ self-dual gauge connections with winding number k is the $[2k+n]\times[2k]$ ADHM matrix

$$\Delta = \Delta_0 - bz = \begin{pmatrix} w \\ a' - z \end{pmatrix} \tag{2.1}$$

with

$$w \equiv (\, J \quad I^\dagger \,) \qquad a' = \begin{pmatrix} B_1 & -B_2^\dagger \\ B_2 & B_1^\dagger \end{pmatrix} \qquad z = \begin{pmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{pmatrix}. \tag{2.2}$$

Here $J, I^\dagger$ and B_i are $[n]\times[k]$ and $[k]\times[k]$ matrices respectively. These matrix elements can be taken to be the coordinates $\mathbf{m} = \{B_1, B_2, I, J\}$ of a $2k^2+2kn$ complex dimensional hyperkähler manifold $\mathbb{C}^{2k^2+2kn}$. z_1, z_2 are the complex coordinates of the euclidean space-time. The form of $b = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ has been fixed by exploiting the symmetries of the ADHM construction. From now on, for the sake of simplicity, anytime we write the matrix z we intend it is multiplied by the $[k]\times[k]$ unit matrix.

The self-dual gauge connection is

$$A_\mu = \bar{U}(x)\partial_\mu U(x) \tag{2.3}$$

with U a $[2k+n]\times[n]$ matrix in the kernel of Δ

$$\bar{\Delta}\, U = 0 = \bar{U}\, \Delta\,. \tag{2.4}$$

Self-duality of the field strength coming from (2.3) requires the matrix Δ to obey the constraint

$$\bar{\Delta}\Delta = f^{-1}_{k\times k}\mathbf{1}_{[2]\times[2]} \tag{2.5}$$

with $f_{k\times k}$ an invertible $[k]\times[k]$ matrix. Substituting (2.1) in (2.5) this condition translates into the so called ADHM constraints

$$f_{\mathbb{C}} = [B_1, B_2] + IJ = 0\,, \tag{2.6}$$

$$f_{\mathbb{R}} = [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + II^\dagger - J^\dagger J = 0\,. \tag{2.7}$$

In the non commutative case in which we allow $[z_1, \bar{z}_1] = -\zeta/2, [z_2, \bar{z}_2] = -\zeta/2$ [14], the above condition becomes $f_{\mathbb{C}} = 0, f_{\mathbb{R}} = \zeta$.

The transformations

$$\begin{pmatrix} w \\ a' - z \end{pmatrix} \to \begin{pmatrix} T_a\, w\, T_\phi^{-1} T_{\epsilon_+} \\ T_\phi T_{\epsilon_-}\,(a' - z)\, T_\phi^{-1} T_{\epsilon_+} \end{pmatrix} \tag{2.8}$$

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with $T_\phi = e^{i\phi} \in \mathrm{U}(k)$, $T_a \in \mathrm{SU}(n)$ and $T_{\epsilon_\pm} = e^{i\epsilon_\pm \sigma_3}$ leave the ADHM constraints (2.7) invariant. The transformations T_ϕ reflect the redundancy in the ADHM description and do not change the gauge connection (2.3). T_a implements global gauge transformations which in the context of $N=2$ SYM can be taken to be diagonal $T_a = \exp\mathrm{diag}(a_1,\ldots,a_n)$. $T_{\epsilon_\pm}$ generate rotations in the complex z_1, z_2 planes with angles $\epsilon_1 = \epsilon_+ + \epsilon_-$ and $\epsilon_2 = \epsilon_+ - \epsilon_-$.

After imposing the $3k^2$ real constraints $f_{\mathbb{C}} = 0, f_{\mathbb{R}} = \zeta$ we are left with a manifold M^ζ of real dimension $k^2 + 4kn$. We define the ADHM manifold $\mathcal{M}_k^\zeta$ as the U(k)-quotient

$$\mathcal{M}_k^\zeta = M^\zeta / U(k) \tag{2.9}$$

with U(k) acting as in (2.8). For generic ζ, $\mathcal{M}_k^\zeta$ is a smooth manifold of dimension $4kn$ [15].

In the absence of v.e.v's, ϵ deformations and in the commutative case, the solutions of the classical Euler-Lagrange equation of motion of $N=2$ SYM for gauginos ψ and scalar field φ can be written as

$$\begin{aligned} \psi(x) &= \bar{U}\left(\mathcal{M} f \bar{b} - b f \bar{\mathcal{M}}\right) U \\ i\varphi_{\mathrm{ferm}}(x) &= \bar{U}\,\mathcal{M} f \bar{\mathcal{M}} U + \bar{U}\begin{pmatrix} 0_{[n]\times[n]} & 0_{[n]\times[2k]} \\ 0_{[2k]\times[n]} & \mathcal{A}'_{f\,[k]\times[k]} \otimes 1_{[2]\times[2]} \end{pmatrix} U\,. \end{aligned} \tag{2.10}$$

The $[k]\times[k]$ matrix $\mathcal{A}'_f = \mathcal{A}'_f$ obeys $\mathbf{L}\mathcal{A}'_f = \Lambda_f$ where

$$\Lambda_f = \frac{1}{2}\left(\bar{\mathcal{M}}\mathcal{M} - (\bar{\mathcal{M}}\mathcal{M})^T\right) \tag{2.11}$$

and the operator $\mathbf{L}$ is defined as

$$\mathbf{L}\cdot\Omega = \{II^\dagger + J^\dagger J, \Omega\} + \sum_{m=1,2} [B_m, [B_m^\dagger, \Omega]] + [B_m^\dagger, [B_m, \Omega]]\,. \tag{2.12}$$

Finally $\mathcal{M} = \left(\begin{smallmatrix} \mu \\ \mathcal{M}' \end{smallmatrix}\right)$ is a constant $[2k+n]\times[2k]$ matrix of grassmanian collective coordinates,the fermionic analogue of the matrix Δ_0 introduced in (2.1). It satisfies the fermionic ADHM constraint

$$\bar{\Delta}_0 \mathcal{M} = \bar{\mathcal{M}}\Delta_0 \tag{2.13}$$

Given two arbitrary variations of the ADHM data (2.2) we define their scalar product as

$$\langle \delta_1\Delta_0, \delta_2\Delta_0\rangle = \mathrm{Tr}_k\,\mathrm{Re}(\delta_1\bar{\Delta}_0 \cdot \delta_2\Delta_0)\,. \tag{2.14}$$

Let us point out that this scalar product (2.14) in the moduli space is induced by the conventional definition of the scalar product of gauge zero modes in euclidean space time. Finally we introduce a U(k)-covariant derivative $D = d + C$ with C determined by the condition that covariant derivatives in the moduli space are orthogonal to all infinitesimal $T_\phi \in \mathrm{U}(k)$ variations

$$0 = \langle D\Delta_0, \delta_\phi\Delta_0\rangle = \mathrm{Tr}_k[\phi(\mathbf{L}\,C - \mathbf{X})] \tag{2.15}$$

with

$$\mathbf{X} = \mathrm{Tr}\,[B_1^\dagger, dB_1] + [B_2^\dagger, dB_2] - dII^\dagger + J^\dagger dJ \;\; - \mathrm{h.c.} \tag{2.16}$$

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It follows from (2.15) that $\mathbf{L}\,C = \mathbf{X}$. The connection C is the analog of the gauge transformation needed to impose the background gauge condition to gauge zero modes. If we put the covariant derivative $D\Delta_0$ in place of $\mathcal{M}$ (2.13), the fermionic constraint is automatically satisfied. This allows to identify fermionic zero modes $\mathcal{M}$ with the one-form $D\Delta_0$ [10, 11]. This can be used to rewrite the fermion bilinear in (2.10) as

$$\bar{U}\,\mathcal{M} f \bar{\mathcal{M}} U = (D\bar{U})(DU) \tag{2.17}$$

with

$$D\bar{U} = -\bar{U}(D\Delta_0) f \bar{\Delta}\,, \qquad DU = -\Delta f (D\Delta_0)^\dagger U\,. \tag{2.18}$$

following[1] from (2.4) [8]. This observation will be helpful in what follows.

2.2 Equivariant forms and the localization formula

In this subsection we briefly review the localization formula. See [17, 18] for a more detailed discussion in this context. We also discuss the geometrical setting in which the localization formalism will be applied. We have seen that M^ζ is acted upon by a Lie group $G \equiv \mathrm{U}(1)^n \times \mathrm{U}(1)^2$ with Lie algebra $\mathfrak{g}$. For every $\xi \in \mathfrak{g}$ we denote by $\xi^* = \xi^s\, T^i_s \frac{\partial}{\partial m^i}$ the *fundamental vector field* associated with ξ, where the ξ^s are the components of ξ in some chosen basis of $\mathfrak{g}$, and the T^i_s are the generators of the action with $\xi^s T^i_s = \delta_{\xi^s}\, m^i$. ξ^* is the vector field that generates the one-parameter group $e^{t\xi}$ of transformations of M^ζ

$$\xi^* \begin{pmatrix} w \\ a' \end{pmatrix} \equiv \delta_\xi \begin{pmatrix} w \\ a' \end{pmatrix} \equiv i_\xi \begin{pmatrix} dw \\ da' \end{pmatrix} = \begin{pmatrix} aw + w\epsilon_+\sigma_3 \\ \epsilon_-\sigma_3 a' + \epsilon_+ a'\sigma_3 \end{pmatrix} \tag{2.19}$$

given by (2.8). We have seen in (2.9) that the moduli space $\mathcal{M}^\zeta_k$ is the space of the $\mathrm{U}(k)$ orbits on M^ζ. In order to make the variations (2.19) orthogonal to these orbits we have to use the previously introduced $\mathrm{U}(k)$ connection C. These orthogonal variations are given by

$$\tilde{\xi}^* \begin{pmatrix} w \\ a' \end{pmatrix} \equiv \delta_{\tilde{\xi}} \begin{pmatrix} w \\ a' \end{pmatrix} \equiv i_\xi \begin{pmatrix} Dw \\ Da' \end{pmatrix} = \begin{pmatrix} aw + w(-\phi + \epsilon_+\sigma_3) \\ [\phi, a'] + \epsilon_-\sigma_3 a' + \epsilon_+ a'\sigma_3 \end{pmatrix} \tag{2.20}$$

with $\phi \equiv i_\xi C$. In order to apply the localization theorem we introduce the equivariant differential d_ξ by letting

$$d_\xi \equiv d + i_\xi\,. \tag{2.21}$$

Acting twice on an equivariant form α (a form satisfying $g\alpha = \alpha$ for all $g \in \mathfrak{g}$) one finds

$$d_\xi^2 \alpha = (d i_\xi + i_\xi d)\alpha = \mathcal{L}_\xi \alpha = 0\,, \tag{2.22}$$

where $\mathcal{L}_\xi$ is the Lie derivative. Thus the space of equivariant forms becomes a differential complex. A form is said to be equivariantly closed if it is equivariant and satisfies $d_\xi\alpha = 0$. The condition that a form is equivariantly closed implies that under certain rather general conditions its top form is exact outside the zeros of ξ^* [9], suggesting that the integral localizes around the critical points.

[1] In (2.18) we have neglected the gauge transformation terms since we think of always dealing with gauge invariant forms.

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It is now time to discuss the action of $N = 2$ SYM on the moduli space. In presence of a v.e.v. for the scalar field, in addition to the piece $8\pi^2 k/g^2$ (g is the gauge coupling constant and k the winding number) the action becomes dependent upon the moduli. It can be evaluated by computing the norm of the zero mode $\mathcal{D}\varphi$, where $\mathcal{D}$ is the covariant derivative with respect to the gauge connection [8, 22]. An alternative representation of S is given as a BRST variation, Q, of a one-form, $\Omega = \langle \xi^*, Dm \rangle$. After identifying Q with $D + i_\xi$ we get $S = d_\xi \Omega$ [10, 11, 19] that implies in particular that $d_\xi S = \mathcal{L}_\xi \Omega = 0$, i.e. e^{-S} is an equivariantly closed form. Moreover at the fixed point $S = 0$.

The equations for the fixed points of the vector field action express the condition that a point of M^ζ acted upon by ξ^* is left invariant up to a U(k) transformation

$$\begin{pmatrix} T_a \, w T_{\epsilon_+} \\ T_{\epsilon_-} \, a' T_{\epsilon_+} \end{pmatrix} = \begin{pmatrix} w \, T_\phi \\ T_\phi^{-1} a' T_\phi \end{pmatrix} . \tag{2.23}$$

This is the same as setting to zero the components of $\tilde{\xi}^*$ in (2.20). Moving along the U(k) orbit (2.23) remains valid for different T_ϕ. For simplicity we choose to solve (2.23) at that point of the orbit where $T_\phi = \mathrm{diag}(e^{i\phi_1}, \ldots, e^{i\phi_k})$

$$\begin{aligned} aJ + J(-\phi + \epsilon_+) &= (\phi_I - a_\alpha - \epsilon_+) J_{\alpha I} = 0 \\ Ia - (\phi + \epsilon_+) I &= (\phi_I - a_\alpha + \epsilon_+) I_{I\alpha} = 0 \\ [\phi, B_\ell] + \epsilon_\ell B_\ell &= (\phi_{IJ} + \epsilon_\ell) B_{\ell JI} = 0 \end{aligned} \tag{2.24}$$

with $\ell = 1, 2$ and $\phi_{IJ} = \phi_I - \phi_J$. Integrals over the ADHM manifold will localize around the ξ^*-fixed points i.e. the solutions of (2.24). The critical points are in one to one correspondence to the partitions of n integers k_α with $\alpha = 1, \ldots, n$ with $\sum_\alpha k_\alpha = k$ the total winding number [13].[2] In the picture of the ADHM construction as a system of k D(-1) branes superposed on n D3 branes, this corresponds to distributing in all possible ways the integer k $D(-1)$-branes between the n D3 branes and then consider all the possible partitions of the resulting k_α 's. $\phi_{I_\alpha} \equiv \phi_{\alpha;(ij)}$ is the U(k)-parameter associated to the box in the i-th row and j-th column in the α-th Young diagram

$$\phi_{I_\alpha} \equiv \phi_{\alpha;(i,j)} = a_\alpha - \epsilon_+ + (i-1)\epsilon_1 + (j-1)\epsilon_2 \tag{2.25}$$

with $i, j \in Y_\alpha$. Solutions to (2.24) are found by setting to zero all components in B_ℓ, I, J except for elements of the form $(B_1)_{(i,j)(i-1,j)}, (B_2)_{(i,j)(i,j-1)}$ and $I_{1,1}, I_{k_1+1,2}, \ldots, I_{k_{n-1}+1,n}$. These elements are later completely fixed by imposing ADHM constraints. This implies in particular that the vector field $\tilde{\xi}^*$ has only isolated zeroes. Assuming that α is equivariantly closed and that $\xi \in \mathfrak{g}$ is such that the vector field ξ^* has only isolated zeroes, x_0, we can state the localization theorem

$$\int_M \alpha = (-2\pi)^{n/2} \sum_{x_0} \frac{\alpha_0(x_0)}{\det^{\frac{1}{2}} \mathcal{L}_{x_0}} , \tag{2.26}$$

[2] It is here assumed that the parameter measuring the non commutativity of the instanton moduli manifold is non zero.

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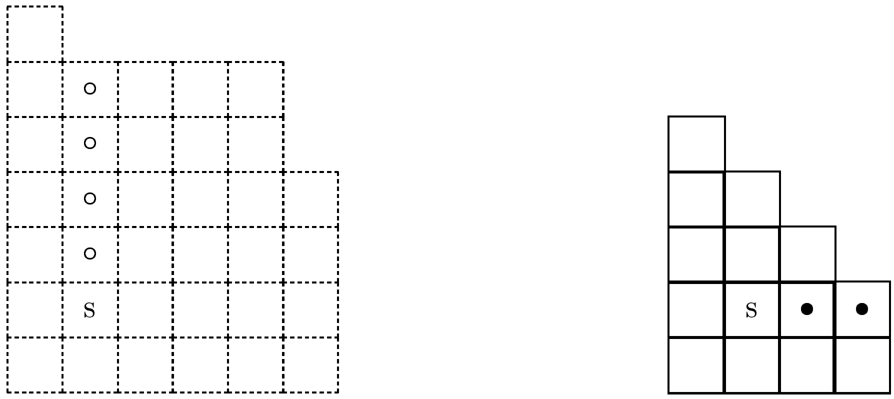

Figure 1: Two generic Young diagrams denoted by Y_α (dotted line) and Y_β (solid line) in the main text.

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where the map $\mathcal{L}_{x_0} : T_{x_0}M \to T_{x_0}M$ is defined as

$$\mathcal{L}_{x_0}(v) = [\xi^*, v] = -\xi^\alpha\, v^i \left(\frac{\partial T^j_\alpha}{\partial m^i}\right)_{x_0} \frac{\partial}{\partial m^j}, \tag{2.27}$$

(which makes sense because at the critical points the components of the fundamental vector field vanish, $\xi^\alpha T^i_\alpha(x_0) = 0$). As first appreciated in [13] this localization takes place in the evaluation of the centered partition function of $N = 2$ SYM. Taking $\alpha = e^{-S}$, with S the multi-instanton action, the application of (2.26) at winding number k leads to [19, 17]

$$\mathcal{Z}_k = \sum_{x_0} \frac{1}{\det \hat{\mathcal{L}}_{x_0}} = \sum_{\{Y_\alpha;\sum_\alpha |Y_\alpha|=k\}} \prod_{\alpha,\beta=1}^{n} \prod_{s\in Y_\alpha} \frac{1}{E_{\alpha\beta}(s)(2\epsilon_+ - E_{\alpha\beta}(s))} \tag{2.28}$$

with

$$E_{\alpha\beta}(s) = a_{\alpha\beta} - \epsilon_1 h_\beta(s) + \epsilon_2(v_\alpha(s) + 1)\,, \tag{2.29}$$

where $h_\beta(s)$ ($v_\alpha(s)$) denotes the horizontal(vertical) distance from the box "s" till the upper (left) end of the $\beta(\alpha)$-diagram i.e. the number of black (white) circles in figure 1. From $\mathcal{Z}_k$ we can build the generating function $\mathcal{Z} = \sum_k \mathcal{Z}_k q^k$, where $q = e^{2\pi i\tau}$ and τ is the coupling of $N = 2$ SYM.

3. The scalar field in presence of a deformation

In this section we will study how the classical solution for the scalar field in (2.10) gets modified in presence of the v.e.v.'s, a_α, and the rotations given from $\epsilon_\pm$. The solution thus found is the general solution to the deformed equations of motion. In the next section we will see how its zero form part can be evaluated at the critical points (2.24).

It is well known that the covariant derivative of the part of the scalar field obeying to the homogeneous Euler-Lagrange equation[3] (turning off the fermionic sources), Z^a_μ, is a

[3] And to the boundary conditions $\lim_{x\to\infty} \varphi = a$.

zero mode satisfying

$$\nabla^\mu Z^a_\mu = 0, \quad \nabla_{[\mu} Z^a_{\nu]} = (\nabla_{[\mu} Z^a_{\nu]})^{dual} \tag{3.1}$$

with

$$Z^a_\mu \equiv 2\,\mathrm{Im}\left\{\bar{U}\tilde{\xi}^*_a \Delta_0 \bar{\sigma}_\mu f \bar{b} U\right\}, \tag{3.2}$$

where $\tilde{\xi}^*_a$ is obtained from (2.20) setting $\epsilon_\pm = 0$. To find the solution, Z_μ, in presence also of the rotations $\epsilon_\pm$, we take as an ansatz (3.2) with $\tilde{\xi}^*_a$ replaced by $\tilde{\xi}^*$ given in (2.20).

With this in mind, following [8, appendix C.1], and writing

$$\xi^*\Delta_0 = \Delta(-\phi + \epsilon_+\sigma_3) + \begin{pmatrix} a & 0 \\ 0 & \phi + \epsilon_-\sigma_3 \end{pmatrix}\Delta - \begin{pmatrix} 0 \\ \epsilon_+ z\sigma_3 + \epsilon_-\sigma_3 z \end{pmatrix} \tag{3.3}$$

one finds

$$\begin{aligned} Z_\mu &= 2\,\mathrm{Im}\left\{\bar{U}\left[\begin{pmatrix} a & 0 \\ 0 & \phi + \epsilon_-\sigma_3 \end{pmatrix}\Delta - \begin{pmatrix} 0 \\ \epsilon_+ z\sigma_3 + \epsilon_-\sigma_3 z \end{pmatrix}\right]\bar{\sigma}_\mu f \bar{b} U\right\} \\ &= 2\,\mathrm{Im}\left\{\bar{U}\mathcal{A}_{\mathrm{bos}}\Delta\bar{\sigma}_\mu f\bar{b}U\right\} - \Omega^\nu_\lambda x^\lambda F_{\mu\nu} = D_\mu \varphi_{\mathrm{bos}} - \Omega^\nu_\lambda x^\lambda F_{\mu\nu} \end{aligned} \tag{3.4}$$

with $\varphi_{\mathrm{bos}} = \bar{U}\,\mathcal{A}_{\mathrm{bos}}\,U$ and

$$\mathcal{A}_{\mathrm{bos}} = \begin{pmatrix} a & 0 \\ 0 & \phi + \epsilon_-\sigma_3 \end{pmatrix}, \qquad \Omega^\mu_\nu = \begin{pmatrix} 0 & -\epsilon_1 & 0 & 0 \\ \epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\epsilon_2 \\ 0 & 0 & \epsilon_2 & 0 \end{pmatrix} \tag{3.5}$$

in agreement with [21]. The complete solution for the scalar field in presence of v.e.v's and gravitational fields is obtained summing φ_{bos} to (2.10), i.e. $\varphi = \varphi_{\mathrm{ferm}} + \varphi_{\mathrm{bos}}$.

Notice that the U(k) field $\phi \equiv i_\xi C$ satisfies the equation

$$\begin{aligned} \mathbf{L}\,\phi &= \mathbf{L}\,i_\xi\,\mathbf{L}^{-1}\mathbf{X} = i_\xi \mathbf{X} \equiv \Lambda_B \\ &\equiv \sum_{\ell=1,2} i\epsilon_\ell\left[B_\ell, B^\dagger_\ell\right] + iJ^\dagger(a+\epsilon_+)J + iI(a+\epsilon_+)I^\dagger \end{aligned} \tag{3.6}$$

and therefore it matches the standard solution at $\epsilon_\pm = 0$.

4. Correlators containing the scalar fields

4.1 The zero form part of the scalar field

We are now ready to discuss the application of (2.26) to the computation of correlators containing scalar fields. The correlator we are interested in is $\langle \mathcal{O}(x)\rangle$ which is a composite of scalar fields. In order to apply the localization formula (2.26) we need to deal with equivariantly closed form. The right objects are $\langle \mathrm{Tr}\,\tilde{\varphi}^m\rangle$ with $\tilde{\varphi} = \tilde{\varphi}_{\mathrm{bos}} + \varphi_{\mathrm{ferm}}$ a closed equivariant form and

$$\tilde{\varphi}_{\mathrm{bos}} = \bar{U}\delta_\xi U\,. \tag{4.1}$$

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That $\tilde{\varphi}$ is equivariantly closed, i.e. $Q\tilde{\varphi} = (D + i_\xi)\tilde{\varphi} = 0$ can be seen as follows.[4] First we recall that in the absence of v.e.v. and gravitational backgrounds the scalar field (2.10) is BRST closed i.e. $D\varphi_{\rm ferm} = 0$. This implies that the only contributions to $Q\varphi_{\rm ferm} = i_\xi \varphi_{\rm ferm}$ comes from fermion bilinears in (2.10) since the contraction i_ξ is trivial on zero forms. Moreover $i_\xi \mathcal{A}'_f = 0$ as follows from the fermionic ADHM constraint (2.13). Finally writing the fermion bilinears as in (2.17) one finds

$$\begin{aligned} Q\tilde{\varphi} &= (D + i_\xi)(\varphi_{\rm ferm} + \tilde{\varphi}_{\rm bos}) = i_\xi(D\bar{U}DU) + D\bar{U}\delta_\xi U + \bar{U}D\delta_\xi U \\ &= \delta_\xi \bar{U}DU - D\bar{U}\delta_\xi U + D\bar{U}\delta_\xi U + \bar{U}D\delta_\xi U = 0 \end{aligned} \tag{4.2}$$

where in the last equation we use $\delta_\xi \bar{U}U = -\bar{U}\delta_\xi U$. We conclude that it is $\tilde{\varphi}$ rather than φ the equivariantly closed form suitable for localization.

The transformation rules for U can be read from those of Δ in (2.8)

$$\begin{aligned} \Delta'(z_\epsilon) &= \begin{pmatrix} T_a & 0 \\ 0 & T_\phi T_{\epsilon_-} \end{pmatrix} \Delta(z) \begin{pmatrix} T_\phi^{-1} T_{\epsilon_+} & 0 \\ 0 & T_\phi^{-1} T_{\epsilon_+} \end{pmatrix} \\ U'(z) &= \begin{pmatrix} T_a & 0 \\ 0 & T_\phi T_{\epsilon_-} \end{pmatrix} U(z_{-\epsilon}) \,. \end{aligned} \tag{4.3}$$

Taking the infinitesimal variation

$$\begin{aligned} \tilde{\varphi}_{\rm bos} = \bar{U}\delta_x i U(z) &= \bar{U} \begin{pmatrix} i a_\lambda & 0 \\ 0 & i\phi + i\epsilon_- \sigma_3 \end{pmatrix} U(z) - \bar{U} i z_\ell \epsilon_\ell \frac{\partial}{\partial z_\ell} U(z) \\ &= \bar{U}\left(\mathcal{A}_{\rm bos} - i z_\ell \epsilon_\ell \frac{\partial}{\partial z_\ell} \right) U(z) = \varphi_{\rm bos} - i\bar{U} z_\ell \epsilon_\ell \frac{\partial}{\partial z_\ell} U(z) \,. \end{aligned} \tag{4.4}$$

It will be shown in the following subsection that it is $\operatorname{Tr}\tilde{\varphi}^2 = \operatorname{Tr}(\tilde{\varphi}_{\rm bos} + \varphi_{\rm ferm})^2$ rather than $\operatorname{Tr}\varphi^2 = (\varphi_{\rm bos} + \varphi_{\rm ferm})^2$ alone the quantity that obeys Matone's relation. Clearly the two quantities coincide in the $\epsilon_\pm \to 0$ limit and therefore the correlators $\langle \operatorname{Tr}\varphi^m \rangle|_{\epsilon=0}$ can be studied using theirs equivariantly closed deformation $\langle \operatorname{Tr}\tilde{\varphi}^m \rangle$.

According to the localization theorem $\langle \operatorname{Tr}\tilde{\varphi}^m \rangle$ reduces to its 0-form part evaluated at the critical points times the inverse determinant appearing in (2.28). The inverse determinant (2.28) was already computed for $N = 2, 2^*, 4$ [13, 19, 17]. In our case, since the action is zero at the critical points, $\alpha_0(x_0)$ is just the sum of the *m-th* powers of the eigenvalues in (4.4) i.e. $\alpha_0(x_0) = \operatorname{Tr}\tilde{\varphi}^m_{\rm bos}$. We remark that at a critical point, the matrix ϕ in (4.4) takes the values (2.25). In fact, substituting (2.24) in the l.h.s. of (3.6) and using (2.12) we obtain an identity.

To end the computation we now have to evaluate U(z). To do so we have to cope with a last problem: to use the localization formula (2.26) the moduli space needs to be compactified and desingularized [15]. This deforms the real ADHM constraint (2.6) into $f_{\mathbb{R}} = \zeta$ as we said earlier. The variables z_1, z_2 become operators in a Fock space $\mathcal{H}$. The non commutative ADHM construction has been previously studied in [14, 23, 24]. In the following we will use the formalism elaborated in [23]. We first remind the reader that

[4] In the previous sections of the paper the forms that enter (2.26) are all gauge invariant. In this case Q coincides with (2.21).

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z_2^4			
z_2^3	$z_1 z_2^3$	$z_1^2 z_2^3$	
z_2^2	$z_1 z_2^2$	$z_1^2 z_2^2$	
z_2	$z_1 z_2$	$z_1^2 z_2$	$z_1^3 z_2$
1	z_1	z_1^2	z_1^3

Figure 2: Notation for a generic Young diagram.

$\mathrm{U}(z)$ is the kernel of the operator matrix $\Delta^\dagger$. Then we rewrite $\mathrm{U}(z)$ as

$$|U(z)\rangle = \begin{pmatrix} |w\rangle \\ |u\rangle \\ |v\rangle \end{pmatrix}, \qquad \begin{array}{c} |w\rangle \equiv w(z_1, z_2)|0,0\rangle \\ |u\rangle \equiv u(z_1, z_2)|0,0\rangle \\ |v\rangle \equiv v(z_1, z_2)|0,0\rangle \end{array} \tag{4.5}$$

$|u\rangle, |v\rangle \in \mathcal{H}^{\oplus k}$, are vectors in $\mathbb{C}^k$ and in the Hilbert space $\mathcal{H}$, $|w\rangle \in \mathcal{H}^{\oplus n}$ is a vector in $\mathbb{C}^n$ and $\mathcal{H}$. Now it is possible to show [25](we will check it later) that the space spanned by (4.5) is isomorphic to the ideal

$$\mathcal{I} = \{w(z_1, z_2)|w(B_1, B_2) = 0\}\,. \tag{4.6}$$

The ideal $\mathcal{I}$ is given by all elements of the form $w(z_1, z_2) = z_1^{k-1} z_2^{l-1}$ with k, l running over boxes outside the Young tableaux Y_α. It is easy to check that B-matrices associated to a tableaux Y_α belong to the ideal (4.6), i.e. $B_1^{k-1} B_2^{l-1} = 0$ for all $k, l \neq Y_\alpha$.[5] If $\mathbb{C}^2[z_1, z_2] = \{z_1^{m-1} z_2^{n-1} | m, n \in \mathbb{Z}_+\}$ is the ring of polynomials of z_1, z_2 and $\mathcal{I} = \{z_1^{k-1} z_2^{l-1} | k, l \neq Y_\lambda\}$ then $Y_\alpha = \mathbb{C}^2(z_1, z_2)/\mathcal{I}$ with $\mathrm{dim} Y_\alpha = k$. For the sake of simplicity we now focus on the $\mathrm{U}(1)$ case since the $\mathrm{U}(n)$ calculation follows straightforwardly from it.

In figure 2 we clarify our notation. The elements of $\mathcal{I}$ are all the monomials that do not appear in the figure and are raised to powers greater than those in the boxes.

We remind the reader that a pair of indices (i, j) is also assigned to each box so that the non-trivial matrix elements in the B-matrices are $(B_1)_{(i,j)(i-1,j)}, (B_2)_{(i,j)(i,j-1)}$. J is always zero while I is a k-dimensional vector whose only element different from zero is $I_1 = \sqrt{k}$.[6] Let us now check the isomorphism between the ideal $\mathcal{I}$ and the space spanned by the states (4.5). From $\Delta^\dagger U = 0$, we find the conditions

$$\begin{aligned} wI - (B_2 - z_2)u + (B_1 - z_1)v &= 0\,, \\ (B_1^\dagger - \bar{z}_1)u + (B_2^\dagger - \bar{z}_2)v &= 0\,. \end{aligned} \tag{4.7}$$

At the critical points, using the explicit expression of w given by (4.6) one is lead to the following form for the (1,1) and generic (i, j) component in the first matrix equation

[5] This can also be easily checked from the explicit solutions for B_1, B_2 given in [19].

[6] The $\mathrm{U}(n)$ case is not much different: J is still zero while I is a $[k] \times [n]$ matrix whose only elements different from zero are $I_{1,1}, I_{k_1+1,2}, \ldots, I_{k_{n-1}+1,n}$. The matrices B_1, B_2 are the same as in the $\mathrm{U}(1)$ case.

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in (4.7)

$$
\begin{aligned}
\sqrt{k} z_1^{k-1} z_2^{l-1} + z_2 u_{(1,1)} - z_1 v_{(1,1)} &= 0\,, \\
(B_2)_{(i,j)(i,j-1)} u_{(i,j-1)} - z_1 v_{(i,j)} + (B_1)_{(i,j)(i-1,j)} v_{(i-1,j)} + z_2 u_{(i,j)} &= 0\,.
\end{aligned} \tag{4.8}
$$

(4.8) is satisfied by $u_{(i,j)} \sim z_1^{k-j} z_2^{l-i-1}, v_{(i,j)} \sim z_1^{k-j-1} z_2^{l-i}$. The indices $i,j \in Y$ $(k,l \notin Y)$ denote those boxes which are inside (outside) a given Young tableaux. The proportionality coefficients of these relations are not important for our subsequent reasonings.[7]

Substituting the form of $\mathrm{U}(z)$ thus obtained in (4.4) we find that the operator δ_ξ acting on $\mathrm{U}(z)$ has eigenvalues $\lambda_i = -ia_\alpha + i(k-1)\epsilon_1 + i(l-1)\epsilon_2$. The Chern character is then defined by the sum $\sum_i e^{i\lambda_i}$ over all eigenvalues λ_i. To this end we introduce the notation $T_1 = e^{i\epsilon_1}, T_2 = e^{i\epsilon_2}, T_{a_\alpha} = e^{ia_\alpha}$. Then

$$
\begin{aligned}
\chi &= \sum_{\alpha=1}^{n} \sum_{(k,l)\notin Y_\alpha} T_{a_\alpha} T_1^{i-1} T_2^{j-1} = \sum_{\alpha=1}^{n} \left(\sum_{(m,n)\in\mathbb{Z}_+^2} T_{a_\alpha} T_1^{m-1} T_2^{n-1} - \sum_{(i,j)\in Y_\alpha} T_{a_\alpha} T_1^{i-1} T_2^{j-1} \right) \\
&= \frac{1}{\mathcal{V}} \left[\sum_{\alpha=1}^{n} T_{a_\alpha} + \sum_{\alpha=1}^{n} \sum_{(i,j)\in Y_\alpha} T_{a_\alpha} T_1^{i-1} T_2^{j-1} (T_1 - 1)(T_2 - 1) \right] \\
&= \frac{1}{\mathcal{V}} \sum_{\alpha=1}^{n} \left\{ e^{ia_\alpha} + \sum_{j_\alpha=1}^{k_{1\alpha}} \left[e^{i[a_\alpha + \epsilon_1(j-1) + \epsilon_2 k'_{j_\alpha}]} - e^{i[a_\alpha - \epsilon_2(j-1)]} - e^{i[a_\alpha + \epsilon_2 j + \epsilon_1 k'_{j_\alpha}]} + e^{i[a_\alpha + \epsilon_2 j]} \right] \right\},
\end{aligned} \tag{4.9}
$$

where k_{j_α} (k'_{j_α}) are the number of boxes in the j-th row (column) of the tableaux Y_α. The sum over the index j_α runs over all the boxes of the first row of the tableaux Y_α. $\mathcal{V}^{-1} = (1-T_1)(1-T_2)$ is a sort of "volume factor".[8] that we will always omit in the following. We stress that the factor $\mathcal{V}^{-1}$ is needed in order to ensure that no negative terms appears in the expansion of the (4.9) so to preserve the interpretation as a Chern character. The numerator in (4.9) reproduces the result in [21]. The 0-form part $\mathrm{Tr}\,\tilde{\varphi}^m_{\mathrm{bos}}$ evaluated at the critical point corresponding to the Young diagrams $\{Y_\alpha\}$, is now given by expanding the exponentials in (4.9) and taking their m-th power

$$
\begin{aligned}
\mathcal{O}_m(\{Y_\alpha\}) &\equiv \mathrm{Tr}\,\tilde{\varphi}^m_{\mathrm{bos}}|_{\{Y_\alpha\}} \\
&= \sum_{\alpha=1}^{n} \left\{ a_\alpha^m + \sum_{j_\alpha=1}^{k_{1\alpha}} \left[[a_\alpha + \epsilon_2(j-1) + \epsilon_1 k'_{j_\alpha} +]^m - [a_\alpha - \epsilon_2(j_\alpha - 1)]^m - \right.\right. \\
&\qquad \left.\left. - [a_\alpha + \epsilon_2 j_\alpha + \epsilon_1 k'_{j_\alpha})]^m + [a_\alpha + \epsilon_2 j_\alpha]^m \right] \right\}.
\end{aligned} \tag{4.10}
$$

Finally

$$
\langle \mathrm{Tr}\,\tilde{\varphi}^m \rangle = \frac{1}{\mathcal{Z}} \sum_{\{k;Y_\alpha\}} \frac{\sum_{\gamma=1}^{n} \mathcal{O}_m(\{Y_\alpha\})}{\prod_{\alpha,\beta}^{n} \prod_{s\in Y_\alpha} E_{\alpha\beta}(s)(2\epsilon_+ - E_{\alpha\beta}(s))} q^k \tag{4.11}
$$

with the sum running over all the sets of n tableaux Y_α with $\sum_\alpha |Y_\alpha| = k$.

[7]The reader can convince himself that these coefficients are zero whenever the exponents of the monomials are negative. To determine these coefficients also the second equation in (4.7) is obviously needed. It is a differential equation for u, v once we put $\bar{z}_{1,2} = \zeta/2\partial/\partial z_{1,2}$.

[8]For small ϵ_1, ϵ_2, $\mathcal{V}^{-1} \sim \epsilon_1\epsilon_2$. We can be cavalier about the renormalization of the prefactor since none of the conclusions of this paper depend on it. A careful study of the renormalization is nevertheless desirable.

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4.2 Matone's relation

In this section we verify that it is $\langle \mathrm{Tr}\,\tilde{\varphi}^2\rangle$ rather than $\langle \mathrm{Tr}\,\varphi^2\rangle$, which satisfies the Matone's relation. As we have already mentioned the two expressions are the same in the limit in which $\epsilon_{1,2} \to 0$, they only differ when gravitational corrections are turned on.

Introducing a coupling τ_m we define

$$e^{-\frac{1}{\epsilon_1\epsilon_2}\mathcal{F}(\tau,\tau_m)} = \int \exp(-S - \tau_m \,\mathrm{Tr}\,\tilde{\varphi}^m)\,. \tag{4.12}$$

Then $\mathcal{Z} = \exp(-\mathcal{F}(\tau,0)/h^2)$. As we already said $\mathrm{Tr}\,\tilde{\varphi}^m_{(0)}|_{\{Y_\alpha\}}$ can be found by expanding the exponential in the numerator of (4.9). The first non-trivial term appears at $m = 2$

$$\mathrm{Tr}\,\tilde{\varphi}^2_{(0)}|_{\{Y_\alpha\}} = a^2 + k\,. \tag{4.13}$$

Only in this case the result does not depends on the shape of the diagram but only on the total winding number k. One finds

$$\begin{aligned}\langle \mathrm{Tr}\,\tilde{\varphi}^2\rangle &\equiv \sum_k \mathcal{G}_k q^k \equiv \frac{1}{\mathcal{Z}}\frac{\partial\mathcal{F}(\tau,\tau_2)}{\partial\tau_2}|_{\tau_2=0} \\ &= \frac{1}{\mathcal{Z}}\sum_k (a^2+\epsilon_1\epsilon_2 k)\,q^k\,\mathcal{Z}_k = a^2 + \epsilon_1\epsilon_2 q\frac{\partial \ln\mathcal{Z}}{\partial q} = a^2 + q\frac{\partial\mathcal{F}(\tau,0)}{\partial q} \\ &= a^2 + \sum_k k\,\mathcal{F}_k\,q^k\,. \end{aligned} \tag{4.14}$$

$\mathcal{G}_k = k\mathcal{F}_k$ is Matone's relation. This result obviously extends to $N = 2^*$ and to the case of matter in the fundamental. The only difference in these cases is the value of the determinants in $\mathcal{Z}_k$. See [17] for their explicit form in these cases. Finally, (4.14) holds at all orders in powers of ϵ_1, ϵ_2 meaning that the Matone's relation should hold once gravitational corrections are taken into account.

4.3 The chiral ring

Here we show that the chiral ring relations are satisfied by $\tilde{\varphi}$. We refer to [6] for notations and detailed explanations. The Coulomb branch of the $\mathrm{U}(n)$ gauge theory with $N = 2$ supersymmetry can be parametrized by the n gauge invariant expectation values

$$u_k = \langle \mathrm{Tr}\,\Phi^k\rangle\,. \tag{4.15}$$

It is known that the expectation values $\langle Tr\Phi^m\rangle$ could be expressed via $u_1, \cdots, u_n$. This relationship could most conveniently be written as

$$\left\langle \mathrm{Tr}\,\frac{1}{z-\Phi}\right\rangle = \frac{P_n'(z)}{\sqrt{P_n(z)^2 - 4q}}\,, \tag{4.16}$$

where $P_N(z)$ is the polynomial defining the quantum Seiberg-Witten curve. (4.16) gives an infinite number of relations, which can be obtained by expanding both sides into powers of $1/z$. In particular, when the gauge group is $\mathrm{SU}(2)$, we have only one independent

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expectation value $u \equiv u_2 = Tr\Phi^2$. In this case (4.16) gives

$$\begin{aligned}\langle \mathrm{Tr}\,\Phi^4\rangle &= 4q + \frac{u^2}{2}\,,\\ \langle \mathrm{Tr}\,\Phi^6\rangle &= 6qu + \frac{u^3}{4}\,,\\ \langle \mathrm{Tr}\,\Phi^8\rangle &= 12q^2 + 6qu^2 + \frac{u^4}{8}\end{aligned} \tag{4.17}$$

Using(4.11) we have computed $\langle \mathrm{Tr}\,\tilde\varphi^m\rangle$ for $m \le 8$ up to 4 instanton contribution.[9] The result reads:

$$\begin{aligned}\langle \mathrm{Tr}\,\tilde\varphi^8\rangle &= 2a^8\Bigg[1 + \frac{14q}{a^4} + \frac{161q^2}{8a^8} + \frac{35q^3}{4a^{12}} + \frac{15337q^4}{2048a^{16}} +\\ &\quad + \left(\frac{105q}{2a^6} + \frac{497q^2}{16a^{10}} + \frac{1505q^3}{64a^{14}} + \frac{99561q^4}{2048a^{18}}\right)(\epsilon_1^2+\epsilon_2^2) +\\ &\quad + \left(\frac{70q}{a^6} + \frac{35q^2}{a^{10}} + \frac{469q^3}{16a^{14}} + \frac{17073q^4}{256a^{18}}\right)\epsilon_1\epsilon_2 + \cdots\Bigg],\\ \langle \mathrm{Tr}\,\tilde\varphi^6\rangle &= 2a^6\Bigg[1 + \frac{15q}{2a^4} + \frac{135q^2}{32a^8} + \frac{125q^3}{64a^{12}} + \frac{16335q^4}{8192a^{16}} +\\ &\quad + \left(\frac{75q}{8a^6} + \frac{135q^2}{64a^{10}} + \frac{735q^3}{128a^{14}} + \frac{124575q^4}{8192a^{18}}\right)(\epsilon_1^2+\epsilon_2^2) +\\ &\quad + \left(\frac{45q}{4a^6} + \frac{75q^2}{32a^{10}} + \frac{525q^3}{64a^{14}} + \frac{92175q^4}{4096a^{18}}\right)\epsilon_1\epsilon_2 + \cdots\Bigg],\\ \langle \mathrm{Tr}\,\tilde\varphi^4\rangle &= 2a^4\Bigg[1 + \frac{3q}{a^4} + \frac{9q^2}{16a^8} + \frac{7q^3}{16a^{12}} + \frac{2145q^4}{4096a^{16}} +\\ &\quad + \left(\frac{q}{4a^6} + \frac{25q^2}{32a^{10}} + \frac{267q^3}{128a^{14}} + \frac{22529q^4}{4096a^{18}}\right)(\epsilon_1^2+\epsilon_2^2) +\\ &\quad + \left(\frac{9q^2}{8a^{10}} - \frac{3q^3}{16a^{14}} + \frac{8535q^4}{1024a^{18}}\right)\epsilon_1\epsilon_2 + \cdots\Bigg],\\ \langle \mathrm{Tr}\,\tilde\varphi^2\rangle &= 2a^2\Bigg[1 + \frac{q}{2a^4} + \frac{5q^2}{32a^8} + \frac{9q^3}{64a^{12}} + \frac{1469q^4}{8192a^{16}} +\\ &\quad + \left(\frac{q}{8a^6} + \frac{21q^2}{64a^{10}} + \frac{55q^3}{64a^{14}} + \frac{18445q^4}{8192a^{18}}\right)(\epsilon_1^2+\epsilon_2^2) +\\ &\quad + \left(\frac{q}{4a^6} + \frac{19q^2}{32a^{10}} + \frac{47q^3}{32a^{14}} + \frac{15151q^4}{4096a^{18}}\right)\epsilon_1\epsilon_2 + \cdots\Bigg].\end{aligned} \tag{4.18}$$

In the limit $\epsilon_{1,2} \to 0$, (4.18) satisfy the ring relations (4.17) up to order q^4. The ϵ-terms represent the first gravitational corrections to the chiral ring. It would be nice to reproduce these deformations of the chiral ring relations from the matrix model perspective (may be along the lines of [26]).[10]

[9] We did this with Mathematica. We have checked higher instantons numbers and powers of the scalar field too. Here we write few sample expressions that can fit the page.

[10] ϵ-terms in $\mathrm{Tr}\,\tilde\varphi^2$ have been shown to match the instanton corrections to gravitational couplings in the $\mathcal{N} = 2$ gauge theory [27].

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A. The commutative limit

In this appendix we discuss the "commutative" limit $\zeta \to 0$ of matrix elements essential for connecting the correlators computed in the non commutative case with those computed in the commutative case. We hope that following this line of reasoning it will be possible to calculate the renormalizing "volume" factor mentioned at the end of section 4.1. This should be needed to establish chiral ring relations in the ϵ deformed case.

Let us suppose that the solution to the equation

$$\bar{\Delta}(\bar{z}_0, z_0) U_{cl}(z_0, \bar{z}_0) = 0 \tag{A.1}$$

in the commutative case ($\zeta = 0$) is known.[11] The function

$$U(z_0, \bar{z}_0|z) = U_{cl}(z_0, \bar{z}_0) e^{\frac{\bar{z}_0 z}{\zeta} - \frac{\bar{z}_0 z}{2\zeta}} \tag{A.2}$$

in the quasi-classical limit $\zeta \to 0$ satisfies the "noncommutative equation"

$$\bar{\Delta}(z, \bar{z}) U(z_0, \bar{z}_0|z) = 0\,, \tag{A.3}$$

in leading order where z and $\bar{z}$ are the non-commutative coordinates, $[\bar{z}, z] = \zeta$ (in the holomorphic representation, the states are holomorphic functions of z and $\bar{z} = \zeta\partial/\partial z$ acts as a derivative). Up to a normalization, the wave function

$$e^{\frac{\bar{z}_0 z}{\zeta} - \frac{\bar{z}_0 z_0}{2\zeta}} \tag{A.4}$$

is the wave packet, centered around the classical variables $(z_0, \bar{z}_0)$, which minimizes Heisenberg's uncertainty relations i.e. a coherent state. Let us calculate the generic matrix element[12]

$$\langle z_0|\varphi|\bar{z}_0'\rangle \equiv \int \bar{U}(\bar{z}_0, z_0|\bar{z}) \mathcal{A} U(z_0', \bar{z}_0'|z) e^{-\frac{z\bar{z}}{\zeta}} \frac{d^4 z}{(\pi\zeta)^2} = \bar{U}_{cl}(\bar{z}_0, z_0) \mathcal{A} U_{cl}(z_0', \bar{z_0}') e^{\frac{\bar{z}_0' z_0}{\zeta} - \frac{\bar{z}_0 z_0 + \bar{z}_0' z_0'}{2\zeta}}\,. \tag{A.5}$$

[11] To simplify the notation, in this appendix $z = (z_1, z_2)$. In agreement with the main text, ζ plays the role of $\hbar$.

[12] Zero modes written in terms of the ADHM variables are always sandwiched between a pair of U(z)'s.

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Now as the simplest example we calculate $\mathrm{Tr}\,\varphi^2$:

$$\begin{aligned}
&\mathrm{Tr}\int \langle z_0|\varphi|\bar{z_0'}\rangle\langle z_0'|\varphi|\bar{z_0}\rangle \frac{d^4z_0'd^4z_0}{(\pi\zeta)^4} = \\
&\quad = \mathrm{Tr}\int \bar{U}_{cl}(\bar{z_0},z_0)\mathcal{A}U_{cl}(z_0',\bar{z_0}')\bar{U}_{cl}(\bar{z_0}',z_0')\mathcal{A}U_{cl}(z_0,\bar{z_0})e^{-\frac{|z_0-z_0'|^2}{\zeta}}\frac{d^4z_0'd^4z_0}{(\pi\zeta)^4} \\
&\quad \approx \mathrm{Tr}\int \bar{U}_{cl}(\bar{z_0},z_0)\mathcal{A}U_{cl}(z_0,\bar{z_0})\bar{U}_{cl}(\bar{z_0},z_0)\mathcal{A}U_{cl}(z_0,\bar{z_0})\frac{d^4z_0}{(\pi\zeta)^2}. \qquad (A.6)
\end{aligned}$$

For small ζ, $\exp -\frac{|z_0-z_0'|^2}{\zeta}/(\pi\zeta)^2$ can be approximated by $\delta^4(z_0-z_0')$. Note that in the first line of the above equation the usual insertions of exponential measure factors are absent since they are already distributed in the wave functions (see (A.4)). So for $\zeta\to 0$ we can approximate expectation values with integrals over the classical variables. A similar result holds also for the higher powers of φ.

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